

# Chiral Magnetic Effect in Hydrodynamic Approximation

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**Abstract.** We review derivations of the chiral magnetic effect (ChME) in hydrodynamic approximation. The reader is assumed to be familiar with the basics of the effect. The main challenge now is to account for the strong interactions between the constituents of the fluid. The main result is that the ChME is not renormalized: in the hydrodynamic approximation it remains the same as for non-interacting chiral fermions moving in an external magnetic field. The key ingredients in the proof are general laws of thermodynamics and the Adler-Bardeen theorem for the chiral anomaly in external electromagnetic fields. The chiral magnetic effect in hydrodynamics represents a macroscopic manifestation of a quantum phenomenon (chiral anomaly). Moreover, one can argue that the current induced by the magnetic field is dissipation free and talk about a kind of "chiral superconductivity". More precise description is a ballistic transport along magnetic field taking place in equilibrium and in absence of a driving force. The basic limitation is exact chiral limit while the temperature—excitingly enough— does not seemingly matter. What is still lacking, is a detailed quantum microscopic picture for the ChME in hydrodynamics. Probably, the chiral currents propagate through lower-dimensional defects, like vortices in superfluid. In case of superfluid, the prediction for the chiral magnetic effect remains unmodified although the emerging dynamical picture differs from the standard one.

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## 1 Introduction

In this chapter <sup>1</sup> we will consider chiral liquids, that is liquids whose constituents are massless fermions. The motivation is an offspring from the discovery of the

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strongly interacting quark-gluon plasma (for review see, e.g., [1]) which is, indeed, build on (nearly) massless quarks. The use of the (relativistic) hydrodynamic approximation is also suggested by the observations on the quark-gluon plasma. Moreover, the state of the chiral liquid is assumed to be asymmetric with respect to the left- and right-fermions. In other words, we concentrate on the case of a non-vanishing chiral chemical potential  $\mu_5$ <sup>2</sup>. The motivation to introduce  $\mu_5 \neq 0$  is rather theoretical than experimental, however, and is rooted in the sphaleron-based picture which predicts that, event-by-event, the plasma is produced as chirally charged [2].

There are a few effects specific for the chiral liquids, the most famous one being the chiral magnetic effect (for review and further references see [3]). By the ChME one understands the phenomenon of induction of electromagnetic current  $\mathbf{j}_{el}$  by applying an external magnetic field  $\mathbf{B}$  to a chiral medium with a non-vanishing  $\mu_5$  :

$$\mathbf{j}_{el} = \frac{q^2 \mu_5}{2\pi^2} \mathbf{B} , \quad (1)$$

where  $q$  is the electric charge of the constituents and  $\mu_5, \mu_5 = (\mu_R - \mu_L)/2$  is the chiral chemical potential. Equation (1) plays a central role in our discussion and can be analyzed from various points of view. An exciting possibility is that the chiral magnetic effect (1) has already been observed in heavy-ion collisions, for a concise review and references see [4]. We will concentrate, however, on the underlying theory rather than on its experimental verification.

Qualitatively, Eq (1) can be understood accounting only for the interaction of spin of quarks with an external magnetic field,  $H_s \sim q(\boldsymbol{\sigma} \cdot \mathbf{B})$ . The overall coefficient in Eq. (1) can readily be found in case of free quarks [5,6,3]. Evaluation of the coefficient requires explicit counting of of zero modes of the Dirac equation for chiral fermions interacting with external magnetic field [6], see also Sect. 3.4. The number of chiral zero modes is controlled by the famous chiral anomaly [7]. Thus, Eq. (1) is a manifestation of the chiral anomaly, as can actually be demonstrated in a number of ways, discussed later.

One of central points is that the chiral magnetic effect can be derived not only in case of non-interacting chiral fermions but also in case of strong interactions between the constituents, provided that the hydrodynamic approximation is granted. There is, though, a change in the Eq (1) which is of pure kinematic nature. Namely, to describe liquid one introduces 4-velocity of an element of the liquid,  $u_\mu(x)$  which is a function of the point  $x$ . The 4-velocity is normalized

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<sup>2</sup> In the realistic QCD case the singlet axial current is anomalous and is not conserved. Therefore, introduction of the chemical potential  $\mu_5$  is rather a subtle issue. In the bulk of the text we ignore this problem concentrating mostly on academic issues. One could have in mind, for example, that the chemical potential  $\mu_5 \neq 0$  is associated in fact with the axial current with isospin  $\Delta I = 1$  which is conserved in the limit of vanishing quark masses. Another possible line of reasoning is to invoke large- $N_c$  limit of Yang-Mills theories. The contribution of the gluon anomaly is then suppressed by large  $N_c$  and the chemical potential  $\mu_5$  can be introduced consistently for the singlet current as well.

such that  $-(u_0)^2 + u_i^2 = -1$ , and in the non-relativistic limit  $u_0 \approx 1$ ,  $u_i \approx v_i$ , where  $i = 1, 2, 3$  and  $\mathbf{v}$  is the 3-velocity entering the hydrodynamic equations in the non-relativistic limit. Eq. (1) is valid now only if the whole of the liquid is at rest. If, on the other hand,  $u_\mu$  is non-trivial Eq. (1) is generalized to

$$(j_\mu)_{el} = \frac{q^2 \mu_5}{2\pi^2} B_\mu, \quad (2)$$

where  $B_\mu \equiv 1/2 \epsilon_{\mu\nu\rho\sigma} u_\nu (\partial_\rho A_\sigma - \partial_\sigma A_\rho)$ ,  $A_\mu$  is the gauge potential of the external electromagnetic field and the chemical potential  $\mu_5$  can depend on the point  $x$ . In the rest frame of an element of the liquid  $u_\mu \equiv (1, 0, 0, 0)$ , and (2) coincides with (1).

Non-renormalization theorems in field theory are quite exceptional and attract a lot of attention. Essentially, there are two best known case studies of non-renormalizability. First, conserved charges are not renormalized, so that, for example, the absolute values of electric charges of electron and proton are the same. And the second example is the non-renormalizability of the chiral anomaly:

$$\partial_\mu j_\mu^5 = \frac{\alpha_{el}}{4\pi} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta}. \quad (3)$$

The absence of higher-order corrections to this anomaly is guaranteed by the Adler-Bardeen theorem [8]. The proof of non-renormalizability of the chiral magnetic effect utilizes both symmetry considerations and the miracle of the perturbative cancellations, revealed by the Adler-Bardeen theorem.

To put the consideration of the chiral magnetic effect into a field-theoretic framework one considers hydrodynamics as a kind of effective field theory, see, e.g., [9]. Viewed as an effective field theory, hydrodynamics reduces to expansion in a number of derivatives from the velocity  $u_\mu$  and thermodynamic quantities. On the microscopic level, hydrodynamics corresponds to the long-wave approximation,

$$l/a \gg 1,$$

where  $l$  is of order of wave length of hydrodynamic excitations and  $a$  is of order distance between constituents.

The hydrodynamic equations reflect symmetries of a dynamical problem considered since they are nothing else but the conservation laws. In the absence of external fields

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j_a^\mu = 0 \quad (4)$$

where  $T_{\mu\nu}$  is the energy-momentum tensor and  $j_a^\mu$  are currents conserved in strong interactions, with the index  $a$  enumerating the currents. Consider liquid at rest and small fluctuations superimposed on it. Generically, fluctuations would be damped down on distances of order  $a$  and do not propagate far away. The exceptions are fluctuations of conserved quantities which cannot disappear and propagate far off. That is why the long-wave, or hydrodynamic approximation reduces to the conservation laws (4).

Explicit form of the hydrodynamic equations (4) depends on how many terms in the gradient expansion are kept in  $T^{\mu\nu}$  and  $j_a^\mu$ . In general,

$$T^{\mu\nu} = wu^\mu u^\nu + Pg^{\mu\nu} + \tau^{\mu\nu} \quad (5)$$

$$j_a^\mu = n_a u^\mu + \nu_a^\mu, \quad (6)$$

where  $w, P, n_a$  are the standard thermodynamical variables, namely, enthalpy,  $w \equiv \epsilon + P$ , pressure and densities of charges. The quantities  $\tau^{\mu\nu}, \nu_a^\mu$  satisfy conditions  $u_\mu \tau^{\mu\nu} = u_\mu \nu_a^\mu = 0$ . In the zeroth order in gradients  $\tau^{\mu\nu} = 0, \nu_a^\mu = 0$ .

The path from relativistic, or chiral hydrodynamics to the anomaly (3) was found first in Ref. [10]. One introduces external electric and magnetic fields so that the current conservation condition is changed into (3). The bridge between a fundamental-theory equation (3) and hydrodynamics is provided then by considering the entropy current  $s_\mu$ . In presence of external fields the standard definition [11] of the current  $s_\mu$  does not ensure the growth of the entropy any longer. To avoid the contradiction with the second law of thermodynamics one includes terms proportional to external fields both into the newly defined entropy current  $s_\mu$  and currents  $j_a^\mu$ . The constraints imposed by the second law of thermodynamics involve the anomaly condition which is not renormalized by strong interactions and turn to be strong enough to (almost uniquely) fix the currents in terms of the anomaly. We reproduce the basic point of this beautiful derivation in Sect. 2.1.

Most recently, it was observed [12,13] (see also [14]) that one can avoid considering the entropy current  $s_\mu$ . Instead, one introduces not only external electromagnetic field but a static gravitational background as well. Equating the hydrodynamic stress tensor and currents (5) to the corresponding structures evaluated at the equilibrium in the gravitational background allows to fix the chiral current and stress tensor without considering the entropy current. This seems to be a very interesting extension of relativistic hydrodynamics. From the perspectives of the present review, derivation [12,13] reveals a novel feature of the chiral magnetic effect. Namely, the corresponding electromagnetic current appears to be non-dissipative since it persists in the equilibrium. We will come back to discuss this point later.

All these derivations of the ChME in fact uncover existence of some other effects as well. In particular, one predicts existence of the chiral vortical effect (ChVE), namely, flow of axial current in the direction of local angular velocity of the liquid,  $j^5 \sim \omega$ . In relativistic covariant notations:

$$\begin{aligned} \delta j_\mu^5 &\approx \frac{\mu^2}{2\pi^2} \omega_\mu \\ \omega_\mu &= \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\nu \partial^\alpha u^\beta, \end{aligned} \quad (7)$$

where  $\omega_\mu$  is the vorticity of the liquid and the chemical potential  $\mu$  is considered to be small. The ChVE was derived first in a holographic set up [15] and is being actively discussed in the literature, along with the chiral magnetic effect. Another example is the axial-vector current  $j^5$  induced by a non-vanishing chemical

potential  $\mu_V$  [16,17]:

$$\mathbf{j}^5 = \frac{q\mu_V}{2\pi^2} \mathbf{B} , \quad (8)$$

which is a kind of parity-reflected companion of Eq. (1).

It is worth emphasizing that all the chiral effects now considered were originally introduced quite long time ago basing on evaluation of loop graphs with non-interacting fermions [18,5]. In particular, it was found in Ref. [18] that rotating a system of non-interacting massless fermions results in a vortical current:

$$\mathbf{j}^5 = \left( \frac{T^2}{12} + \frac{\mu^2}{4\pi^2} \right) \mathbf{\Omega} , \quad (9)$$

where  $\mathbf{\Omega}$  is the angular velocity of the rotation. In the term proportional to  $\mu^2$  one readily recognizes Eq. (7) above<sup>3</sup>. It took many years, however, to prove that the result (9) is essentially not modified by strong interactions. As is mentioned above, the origin of the  $\mu^2$  in Eq. (9) term can be traced back to the chiral anomaly and it is not renormalizable.

The status of radiative corrections to the  $T^2$  term in Eq. (9) has been clarified only very recently [19,20]. First, one relates the chiral vortical effect to a static correlator of axial current and of momentum density. As far as only fermionic part is kept in the operator of momentum density, all the higher-order contributions to the  $T^2$  term cancel and result (9) remains valid. The proof is based on analysis of Feynman graphs and echoes the proof [21] of non-renormalizability of the topological mass of a gauge field in 3d gauge theory. There is, however, a gluonic part of the momentum density and it generates a calculable two-loop correction to (9). We reproduce the basic points of the proof following [19] in Sect. 2.3.

Reference to anomalies of the fundamental theory which arise due to weak coupling to external fields (electromagnetic or gravitational) can be avoided by applying an effective field theory. This effective field theory elevates chemical potentials to interaction constants, see [22,23,24] and references therein. The corresponding vertices can be obtained from the standard electromagnetic interaction by substitution

$$qA_\mu \rightarrow \mu u_\mu , \quad (10)$$

where  $\mu$  is the chemical potential associated with the conserved charge  $q$ . The effective theory is anomalous and does reproduce through these anomalies the chiral magnetic effect and the  $\mu^2$  term in the chiral vortical effect, see Eq. (9).

<sup>4</sup>. We will give further details in Sect. 2.4.

Non-renormalizability commonly implies topological nature of the corresponding term. Moreover, if the currents are topological they are non-dissipative. The

<sup>3</sup> To compare (9) and (7) one should keep in mind that in notations of Ref. [18] the current  $\mathbf{j}$  in Eq (9) is the current of right-handed fermions alone and, thus, constitutes one half of the chiral current entering Eq. (7).

<sup>4</sup> Note that in the underlying fundamental field theory there are no anomalies associated with non-vanishing chemical potential  $\mu$ . This observation is in no contradiction with the fact that such anomalies do arise in the language of the effective theory.

best known example of such a type is provided by the integer quantum Hall effect (for the background and review see, e.g., [25]). Consider a two-dimensional system with external electric field  $\mathbf{E} = (E_1, 0)$  applied. Then there arises electric current  $j_i$  such that

$$j_i = \sigma_{ik} E_k ,$$

where  $\sigma_{ik}$  are coefficients. The integer quantum Hall effect is characterized by a non-diagonal  $\sigma_{12} \neq 0$  and  $\sigma_{11} = 0$ :

$$\sigma_{12} = \nu \frac{e^2}{h} , \quad (11)$$

where  $\nu$  is integer. The work produced by the external electric field is equal to the product  $W = j_i E_i$ . The Hall current is clearly not associated with any work done and this observation suggests strongly that the Hall current is dissipation-less. For further examples of this type see, e.g., [26].

Since the chiral magnetic current (1) is not associated either with any work done by an external field it seems natural to assume that the chiral magnetic current is also dissipation-less. This suggestion is made first in Ref [27] basing on somewhat different arguments, see Sect. 3.1.

Now we come to a question, however, which has not been answered yet. Namely, topological, or dissipation-less currents usually manifest existence of a macroscopic quantum state. Well known examples are the superfluidity of weakly interacting Bose-liquid or the same Hall current (11). In these two cases the nature of the macroscopic quantum states is well understood. Also, in case of non-interacting fermions topological nature of the chiral magnetic effect has been demonstrated first long time ago [6]. Claiming the chiral magnetic current (1) to be topological in hydrodynamic approximation as well we imply quantum nature of the corresponding ground state.

This problem of explicit constructing the quantum state can be addressed in some more detail within approach which starts with a microscopical picture and the central role is played then by low-dimensional defects. A well known example of this kind is provided by vortices in rotating superfluid. In more abstract language, this approach goes back also to papers [28,6,29]. It was demonstrated that defects in field theory are closely tied to the realization of anomaly. In particular, it was shown in [29] that anomaly in  $2n+2$  dimensional theory is connected with  $2n$  dimensional index density and can be understood in terms of fermion zero modes on strings and domain walls. In all the cases the chiral current is carried by fermionic zero modes living on the defects.

One can expect, therefore, that anomaly in effective, hydrodynamic theory is realized in chiral superfluid system on vortex-like defects. The continuum-medium results (7), (1) can arise then upon averaging over a large number of defects. In case of the chiral magnetic effect such a mechanism was considered, in particular, in Refs. [17], [2] and the final result (1) is reproduced on the microscopic level as well. The vortices considered in [17], [2] are simply the regions of space free of the medium substance. In case of superfluidity the vortices are better understood and the microscopical picture for the chiral effects can be

clarified to some extent [30]. The outcome of the analysis in terms of defects, or vortices is that the chiral magnetic effect does survive without any change. As for the vortical chiral effect it is modified in the capillary picture by a factor of two:

$$\left(\delta J_\mu^5\right)_{capillary} = 2\frac{\mu^2}{2\pi^2}\omega_\mu \quad (12)$$

We will give details in Sect. 3.4.

Upon introducing the reader to the topics to be discussed in this review, we would like to emphasize that there are many other interesting results which could have been included into the review but are actually not covered. The reason is mostly to avoid too much overlap with other chapters of this volume. A notable example of this kind is the holographic approach to the ChME which is reviewed, in particular, in Ref. [31]. The same remark applies to the phenomenological manifestations of the gravitational anomaly. Finally, there are very interesting applications of the technique used to condensed-matter systems. However, reviewing these applications goes beyond the scope of the present notes.

To summarize, we concentrate on two basic issues, non-renormalizability and dissipation-free nature of the chiral magnetic and chiral vortical effects. In Sect. 2 we consider non-renormalization theorems within various approaches outlined above (thermodynamic, geometric, diagrammatic, effective field theories). The derivations of the theorems make it also clear that the chiral effects considered are dissipation free. In Sect. 3 we review further arguments in favor of dissipation-free nature of the chiral effects. In this section we also introduce a microscopic picture in terms of defects of lower dimensions. Sect. 4 is conclusions.

## 2 Non-renormalization Theorems

### 2.1 Non-renormalization theorems in thermodynamic approach

In this subsection we reproduce the basic steps of the pioneering derivation [10] of chiral effects, the chiral vortical effect first of all, which utilizes only the chiral anomaly in external electromagnetic fields, thermodynamics and hydrodynamic approximation. To simplify the algebra, we consider first a single conserved current, chiral at that. Moreover, we consider the chiral symmetry not spontaneously broken (otherwise, we should have modified hydrodynamics)

In presence of external electromagnetic fields both the energy-momentum tensor and chiral current are not conserved any longer. The current is not conserved because of the anomaly, while the energy is not conserved because external electric field executes work on the system. Thus, one starts with the equations:

$$\partial_\mu j^\mu = CE^\mu B_\mu, \quad (13)$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda \quad (14)$$

where  $C$  is the coefficient determined by the anomaly (e.g. for QED  $C = \frac{1}{2\pi^2}$ ). Turning to the thermodynamics, we have to introduce, following textbooks [11],

an entropy current  $s_\mu$  consistent with the second law of thermodynamics. In the approximation of an ideal liquid the condition is  $\partial_\mu s^\mu = 0$ .

There are no general rules to construct  $s_\mu$ . As a first guess one can try  $s_\mu = su_\mu$  where  $s$  is the entropy density. Moreover, put the gradient terms  $\nu^\mu, \tau^{\nu\mu} = 0$  for simplicity. However, using the equations (13) and

$$dP = sdT + nd\mu$$

where  $\mu$  is chemical potential, one readily derives

$$\partial_\mu(su^\mu) = -C\frac{\mu}{T}E \cdot B \quad (\nu^\mu, \tau^{\nu\mu} = 0). \quad (15)$$

The right-hand side of this equation does not have a definite sign and, therefore, one cannot accept  $su^\mu$  as a definition of the entropy current in presence of the anomaly. Thus, we should continue with our guess-work to construct the entropy current. Note that it is quite a common situation. For example, consider non-ideal liquid with non-zero  $\nu^\mu$  (and  $\tau^{\mu\nu} = 0$ ). Then one has to modify [11] the entropy current defining it as  $s^\mu = su^\mu - \frac{\mu}{T}\nu^\mu$  so that for the newly defined entropy current  $\partial_\mu s^\mu = 0$ .

In presence of the chiral anomaly we can use the same idea and redefine the entropy current [10] by introducing terms proportional to the magnetic field and vorticity. To simplify equations we will not account for dissipative terms, viscosities and electrical conductance. One can check that inclusion of these terms does not change the result [10]. Moreover, in the next subsection we will see that there are general reasons for the dissipative terms to be actually not relevant. Thus, expanding in the fields we look for solution for the matter current of the form:

$$\begin{aligned} j_\mu &= nu_\mu + \nu_\mu \\ \nu_\mu &= \xi_\omega \omega_\mu + \xi_B B_\mu, \end{aligned} \quad (16)$$

where  $\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu\partial^\alpha u^\beta$  is the vorticity,  $B_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$  is the magnetic field in the rest frame of liquid element (electric field  $E_\mu = F_{\mu\nu}u^\nu$ ) and  $\xi_\omega, \xi_B$  are unknown functions of thermodynamic variables. For the entropy current, we assume:

$$s_\mu = su_\mu - \frac{\mu}{T}\nu_\mu + D_\omega \omega_\mu + D_B B_\mu, \quad (17)$$

where  $D_\omega, D_B$  are further unknown functions.

Conservation of the entropy current now reads:

$$\partial_\mu(D_\omega \omega^\mu) + \partial_\mu(D_B B^\mu) - \nu^\mu \left( \partial_\mu \frac{\mu}{T} - \frac{\mu}{T} \right) - C\frac{\mu}{T}E \cdot B = 0. \quad (18)$$

For the ideal liquid ( $\tau^{\mu\nu} = 0$ ) the following identities hold:

$$\partial_\mu \omega^\mu = -\frac{2}{w} \omega^\mu (\partial_\mu P - nE_\mu) \quad (19)$$

$$\partial_\mu B^\mu = -2\omega \cdot E + \frac{1}{w} (-B \cdot \partial P + nE \cdot B). \quad (20)$$



Moreover, the coefficients in front of independent kinematical structures  $\omega^\mu, B^\mu, E \cdot \omega, E \cdot B$  in relation (18) should vanish:

$$\begin{aligned} \partial_\mu D_\omega - 2 \frac{\partial_\mu P}{\epsilon + p} D_\omega - \xi_\omega \partial_\mu \frac{\mu}{T} &= 0 \\ \partial_\mu D_B - \frac{\partial_\mu P}{\epsilon + p} D_B - \xi_B \partial_\mu \frac{\mu}{T} &= 0 \\ \frac{2n D_\omega}{\epsilon + p} - 2D_B + \frac{\xi_\omega}{T} &= 0 \\ \frac{n D_B}{\epsilon + p} + \frac{\xi_B}{T} - C \frac{\mu}{T} &= 0. \end{aligned} \quad (21)$$

To proceed further one has to choose a basis of thermodynamic variables and it is convenient to take this basis as  $(P, \tilde{\mu} = \frac{\mu}{T})$ . The thermodynamic derivatives in the basis look as  $(\frac{\partial T}{\partial P})_{\tilde{\mu}} = \frac{T}{w}$ ,  $(\frac{\partial T}{\partial \tilde{\mu}})_{\tilde{P}} = -\frac{nT^2}{w}$ . Since the thermodynamic gradients  $\partial_\mu P, \partial_\mu \tilde{\mu}$  are independent the first two equations in (21) imply four conditions:

$$\begin{aligned} -\xi_\omega + \frac{\partial D_\omega}{\partial \tilde{\mu}} &= 0, & -\xi_B + \frac{\partial D_B}{\partial \tilde{\mu}} &= 0 \\ \frac{\partial D_\omega}{\partial P} - \frac{2}{w} D_\omega &= 0, & \frac{\partial D_B}{\partial P} - \frac{1}{w} D_B &= 0, \end{aligned} \quad (22)$$

and the general solution for  $D_\omega, D_B$  looks as

$$D_\omega = T^2 d_\omega(\tilde{\mu}), \quad D_B = T d_B(\tilde{\mu}),$$

with functions  $d_\omega, d_B$  being so far arbitrary. Then the last two equations in (21) reduce to simple differential equations which can be readily solved. As a result, the functions  $d_\omega(\tilde{\mu}), d_B(\tilde{\mu})$  get fixed, up to the integration constants [33,23] which are the values of the functions at  $\tilde{\mu} = 0$ .

For the chiral kinetic coefficients we finally obtain:

$$\xi_\omega = C\mu^2 \left(1 - \frac{2}{3} \frac{\mu \cdot n}{\epsilon + p}\right) + C_\omega T^2 \left(1 - \frac{\mu \cdot n}{\epsilon + p}\right) \quad (23)$$

and

$$\xi_B = C \left(\mu - \frac{1}{2} \frac{\mu^2 n}{\epsilon + p}\right), \quad (24)$$

where the constant  $C$  determines the anomaly and is fixed while the constant  $C_\omega$  remains undetermined. We will come back to evaluate  $C_\omega$  in Sect. 2.3 following the paper [19]. Note that we omitted a similar constant of integration from (24). The reason is that such a constant can appear in fact only due to parity-violating interactions [33] and, having in mind eventual applications to parity-conserving theories, we suppressed it right away.

The non-vanishing  $\xi_\omega, \xi_B$  exhibit what we call chiral vortical and chiral magnetic effects, respectively. At temperature  $T = 0$  the functions  $\xi_\omega(\mu), \xi_B(\mu)$  are

fixed in terms of the coefficient  $C$  which can be read off from the chiral anomaly and is not renormalized by strong interactions. This is the content of the non-renormalization theorem of the chiral effects in hydrodynamic approximation. It is worth emphasizing that if parity is conserved the chiral magnetic and vortical effects are manifested in fact in different currents, axial and vector, respectively. On the other hand, according to Eq (43), they appear in one and the same current. The reason is that through postulating conservation of a single chiral current we actually admitted for a parity violating "strong interaction". The case of a few currents, which allows for parity-conserving strong interactions was considered, in particular in Refs [33,23]. The outcome of the calculation is essentially the same: the chiral effects are fixed in all the currents, up to constants of integration.

## 2.2 Non-renormalization theorems in geometric approach

Let us recall the reader one of simplest derivations of the chiral magnetic effect [6,2]. The anomaly,

$$\partial^\mu j_\mu^5 = \frac{q^2}{2\pi^2} \mathbf{E} \cdot \mathbf{B} \quad (25)$$

can be rewritten as an equation for the production rate of chiral particles. Denoting the total chirality as  $N_5$ , where  $N_5 \equiv N_R - N_L$  and  $N_R(N_L)$  is the number of right- (left-) handed particles we have:

$$\frac{dN_5}{dt d^3x} = \frac{q^2}{2\pi^2} \mathbf{E} \cdot \mathbf{B} . \quad (26)$$

Production of particles requires for energy to be deposit into the system. The source of this energy is the work done by the external electric field. Therefore:

$$\int d^3x \mathbf{j}_{el} \cdot \mathbf{E} = \mu_5 \frac{dN_5}{dt} = \frac{q^2 \mu_5}{2\pi^2} \int d^3x \mathbf{B} \cdot \mathbf{E} \quad (27)$$

where  $\mathbf{j}_{el}$  is the electric current and  $\mu_5$  is the energy needed to produce a particle. Tending  $\mathbf{E} \rightarrow 0$  we learn from Eq. (27) that there survives a non-vanishing current in this limit:  $\mathbf{j}_{el} = (q^2 \mu_5 / 2\pi^2) \mathbf{B}$ , and we come back to Eq. (1) As is mentioned above, the current is non-dissipative since magnetic field does not produce any work. One can also say that the current  $\mathbf{j}_{el}$  exists in the equilibrium.

To summarize, one can calculate the non-dissipative current associated with the magnetic field by introducing electric field, taking the system out of the equilibrium in this way and then tending the electric field back to zero. A similar technique is commonly applied to study spontaneous symmetry breaking.

Recently it has been realized [12,13] that the procedure can be generalized in a rather unexpected way. Namely, one introduces not only external electromagnetic field but static gravitational field as well and studies equilibrium in this background. All the terms in chiral currents and energy-momentum tensor are fixed in the equilibrium and are non-dissipative. Eventually one can go back to the flat space.

The basic object in the approach [12,13] is the generating functional  $W$  as a function of external electromagnetic and gravitational fields, or sources,

$$W = \int d^d x L(\text{sources}(x)) ,$$

where  $W = \ln Z$  and  $Z$  is the partition function. Differentiating  $W$  with respect to the sources one evaluates in the standard way energy-momentum tensor and currents, as well as and their correlators in the equilibrium. In particular,

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}} , \quad \langle j_\mu \rangle = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta A_\mu} . \quad (28)$$

In the spirit of the hydrodynamic approximation, one expands  $W$  in the number of derivatives, both from the sources and thermodynamic variables and reiterates the procedure in each order in the expansion.

The medium is characterized by the time-like vector  $u^\mu$  which in the zeroth order in the number of derivatives can be chosen as  $u^\mu \sim (1, 0, 0, 0)$ . Apart from  $u^\mu$  the generating functional can depend on observables that are local in space but non-local in Euclidean time. In the zeroth order in derivatives, the invariant length  $L$  of the time circle is one of such observables. Also, there is Polyakov loops  $P_A$  of any U(1) gauge fields. Therefore, temperature  $T$  and chemical potential  $\mu$  are defined geometrically as

$$T = 1/L , \quad \mu = \ln P_A / L . \quad (29)$$

These are simplest examples of diffeomorphic and gauge invariant scalars.

We pause here to emphasize that the outcome of the calculation are static correlators which are the same in the Euclidean and Minkowskian versions of the theory. Therefore the relations obtained are in fact thermodynamic in nature. Static correlators are to be distinguished from correlators which determine, through the Kubo formula, such transport coefficients as viscosity. In the latter case one considers correlator of certain components of the stress tensor at momentum transfer  $\mathbf{q} \equiv 0$  and frequency  $\omega \rightarrow 0$ . On the other hand, correlators considered here correspond to  $\omega \equiv 0$ ,  $\mathbf{q} \rightarrow 0$ . This, subtle at first sight, difference is crucial for continuation to the Euclidean space. It is straightforward to realize that the chiral magnetic effect for a time-independent magnetic field is indeed determined by a static correlator of components of electromagnetic current, see, e.g., [34,35]. To demonstrate this, it is convenient to begin with the standard Kubo relation for electric conductance:

$$\sigma_E = \lim_{\omega \rightarrow 0} \frac{i}{\omega} \langle j_i, j_i \rangle |_{\mathbf{q}=0} \quad (30)$$

where  $\sigma_E$  determines the electric current in terms of a time-independent electric field,  $\mathbf{j}_{el} = \sigma_E \mathbf{E}$ , and  $\langle (j_{el})_i, (j_{el})_i \rangle$  is the retarded correlator of the components of the electromagnetic current (with no summation over the index

*i*). Since both electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$  are related to the same vector-potential  $\mathbf{A}$  ( $\mathbf{E} = -i\omega\mathbf{A}$ ,  $\mathbf{B} = i\mathbf{q} \times \mathbf{A}$ ) one concludes:

$$\sigma_B = \lim_{q_n \rightarrow 0} \sum_{ij} \epsilon_{ijn} \frac{i}{2q_n} \langle (j_{el})_i, (j_{el})_j \rangle |_{\omega=0}, \quad (31)$$

where  $\sigma_B$  is defined as  $\mathbf{j}_{el} = \sigma_B \mathbf{B}$ . Thus, it is indeed a static correlator which we need to evaluate the ChME.

Probably, the best known example of the use of static correlators is the generation of photon screening mass through the Higgs mechanism. Namely, the correlator of components of electromagnetic current in superconducting case looks as:

$$\lim_{\mathbf{q} \rightarrow 0} \int d^3x \exp(i\mathbf{q} \cdot \mathbf{r}) \langle j_i(\mathbf{x}), j_k(0) \rangle \sim (\delta_{ik} - \mathbf{q}_i \mathbf{q}_k / \mathbf{q}^2),$$

where presence of a pole signals superconductivity while the local term proportional to  $\delta_{ik}$  signifies a non-vanishing photon mass. A similar role of a signature of superfluidity is played by a pole in the static correlator of the components of momentum density:

$$\lim_{\mathbf{q} \rightarrow 0} \int d^3x \exp(i\mathbf{q} \cdot \mathbf{r}) \langle T_{0i}(\mathbf{x}), T_{0k}(0) \rangle \sim \mathbf{q}_i \mathbf{q}_k / \mathbf{q}^2. \quad (32)$$

Note, however, absence of the local term proportional to  $\delta_{ik}$ . This result is readily understood if we start from considering non-trivial gravitational background. The local terms are associated then with covariant derivatives, say,  $D_i v_k = \partial_i v_k + \Gamma_{ik}^l v_l$ , where  $v_i$  is a vector and  $\Gamma_{ik}^l$  are the Christoffel symbols. The  $\Gamma_{ik}^l$  symbols contain only derivatives from the components of the metric tensor  $g_{\mu\nu}$  and we immediately conclude that there could be no  $\delta_{ik}$  local term in the correlator of  $T_{0i}$  components. Thus, we see that introducing first gravitational background does allow to fix the subtraction term in a static correlator in the flat space.

Currents which we are considering now are somewhat similar to the standard superfluid current [44]. But there are important differences as well. In particular, the value of the superfluid current is not fixed in the equilibrium while the current associated with the ChME has a unique value. Now, the statement is [12,13] that all the non-dissipative pieces in  $\langle j_\mu \rangle$ ,  $\langle T_{\mu\nu} \rangle$  can be conveniently determined by embedding the system into static electromagnetic plus gravitational background. We will outline briefly the proof following [12].

Static gravitational background in all the generality can be parameterized as follows:

$$ds^2 = -e^{2\sigma(x)} (dt + a_i(x) dx^i)^2 + g_{ij}(x) dx^i dx^j, \quad (33)$$

where  $x_i$  are spatial coordinates ( $i = 1, 2, 3$  for definiteness),  $\partial_t$  is the Killing vector on this manifold, gravitational potentials  $\sigma, a_i, g_{ij}$  are smooth functions of the coordinates  $x_i$ . One assumes also presence of a static U(1) gauge field  $A$ ,

$$A = A_0 dx^0 + A_i dx^i \quad (34)$$

The  $A_0$  component is related to the chemical potential, see Eq (29).

Consider first zeroth order in gradients. Then it is quite obvious that in the equilibrium

$$u_{(0)}^\mu = e^{-\sigma(x)}(1, 0, 0, 0), \quad T_{(0)}(x) = e^{-\sigma(x)}T_0, \quad \mu_{(0)}(x) = e^{-\sigma(x)}A_0, \quad (35)$$

where  $\sigma(x)$  enters the metric (33) and subscript (0) refers to the zeroth order in expansion in derivatives. Indeed, expressions for  $T_{(0)}(x), \mu_{(0)}(x)$  can be obtained directly from their invariant definition (29) while  $u_{(0)}^\mu(x)$  is fixed by the normalization condition. Thus, the function  $W$  to this lowest order reduces to

$$W_{(0)} = \int \sqrt{g_3} \frac{e^\sigma}{T_0} P(T_0 e^{-\sigma}, A_0 e^{-\sigma}), \quad (36)$$

where  $P(T, \mu)$  is pressure as function of temperature and chemical potential in flat space.

In higher orders in derivatives one expands  $u^\mu, T, \mu$  further:

$$u^\mu = u_{(0)}^\mu + u_{(1)}^\mu + u_{(2)}^\mu + \dots, \quad (37)$$

$$T = T_{(0)} + T_{(1)} + T_{(2)} + \dots, \quad (38)$$

$$\mu = \mu_{(0)} + \mu_{(1)} + \mu_{(2)} + \dots, \quad (39)$$

where  $u_{(n)}^\mu, T_{(n)}, \mu_{(n)}$  are expressions of  $n$ -th order in derivatives acting on the background fields  $\sigma, A_0, A_i, g_{ij}$ . It is important that both  $T_{(n)}$  and  $\mu_{(n)}$  are constructed on the same set of gauge- and diffeomorphic-invariant scalars. Let us denote the number of such scalars as  $s_{(n)}$ . As for the four velocity  $u^\mu$  normalized to unit,  $\delta u^0$  can be expressed in terms of  $\delta u^i$  and is not independent. Variations of the vector  $u_{(n)}^i$  are expanded in the set of independent invariant vector combinations. The total number of such combinations is denoted as  $v_{(n)}$ .

Consider now a general decomposition of the energy-momentum tensor and of the current  $j_\mu$ :

$$T^{\mu\nu} = E u^\mu u^\nu + P \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \tilde{\tau}^{\mu\nu}, \quad (40)$$

$$j^\mu = N u^\mu + \nu^\mu \quad (41)$$

where  $q^\mu u_\mu = \tilde{\tau}^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\mu = 0$ ,  $g_{\mu\nu} \tilde{\tau}^{\mu\nu} = 0$ . Similar to the observations above, the scalars  $E, P, N$  can be written as expansions in independent gauge- and diffeomorphic-invariant scalars, vector  $q^\mu$  is expanded in independent SO(3) vector structures while expansion of  $\tau^{\mu\nu}$  would require introduction of a set of independent SO(3) tensor structures, with trace zero. The total number of the tensor structures is denoted by  $t_{(n)}$ . Note that within the standard relativistic hydrodynamics the energy-momentum tensor is defined somewhat different, in the so called Landau gauge, see Eq (5). Namely, there is no vector  $q^\mu$  and  $\epsilon, P, n$  are those functions of temperature  $T$  and of chemical potential as determined by flat-space equilibrium thermodynamics. As a result, it is only  $\tau_{\mu\nu}$  and  $\nu^\mu$  which are to be expanded in gauge and diffeomorphic invariant structures introduced

above. Therefore, the hydrodynamic tensors (5) are expanded in  $s_{(n)} + v_{(n)} + t_{(n)}$  invariant structures.

The central point of the procedure invented in [12,13] is to equate the standard hydrodynamic tensors (5) to the expressions obtained by differentiating the functional  $W$ , see Eq (28). This can be done only at the equilibrium point. In the equilibrium, not all the gauge- and diffeomorphic-invariant structures introduced above survive. The number of non-vanishing structures is called  $s_{(n)}^e, v_{(n)}^e, t_{(n)}^e$ . Moreover, these, surviving terms are non-dissipative since they exist in the equilibrium. As for the dissipative terms, which correspond to the structures vanishing in the equilibrium one gets no predictions or constraints concerning them.

By simple counting, the total number of the constraints is  $3s_{(n)}^e + 2v_{(n)}^e + t_{(n)}^e$  and this number is exactly the same as needed to both find expansion corrections of the  $n$ -th order to  $T, \mu, u^i$  and determine the equilibrium stress tensor and current in the Landau gauge (5)<sup>5</sup>. Indeed, the expansion of  $T, \mu, u^i$  at the equilibrium point depends on  $2s_{(n)}^e + v_{(n)}^e$  parameters while expansion of the energy-momentum tensor and of the current in the Landau gauge brings in  $s_{(n)}^e + v_{(n)}^e + t_{(n)}^e$  terms. This completes the general proof that all the non-dissipative terms, like the chiral magnetic effect, are fixed uniquely. Actual details and explicit examples can be found in Refs [12,13]. In particular, the results of Ref [10] have been reproduced.

The method of Refs. [12,13] outlined above allows to derive systematically chiral effects, like the ChME, for any number of conserved and anomalous currents and to any order in the derivative expansion. Remarkably, one avoids considering the entropy current  $s_\mu$  altogether. The derivation makes it clear that one can fix only currents existing in the equilibrium. In other words, the currents are non-dissipative [35]. The issue will become central for us in Section 3.

### 2.3 Non-renormalization theorems in diagrammatic approach

Very recently, there was a remarkable development [19,36] in understanding the temperature-dependent chiral vortical effect, see the  $T^2$  term in Eqs. (9), (23). Namely, it was demonstrated that the bare-loop result (9) is modified in two-loop order in a well defined way.

As a preliminary remark, let us notice that the chiral vortical effect is determined [35] in terms of a static correlator, similar to the case of the chiral magnetic effect, see Eq. (31). We recall the reader that the chiral vortical effect is defined in terms of the coefficient  $\xi_\omega$ , see Eq. (43). In the non-relativistic limit we have the following piece in the axial-vector  $j_i^5$ :

$$\delta(j_i^5) = \xi_\omega \epsilon_{ijk} \partial_j v_k, \quad (42)$$

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<sup>5</sup> Alternatively, this counting can be considered as a proof of the possibility to introduce the Landau gauge (5).

where  $v_k$  are components of the 3-velocity of an element of the liquid. Then for the coefficient  $\xi_\omega$  one gets [35]:

$$\xi_\omega = \lim_{q_n \rightarrow 0} \sum_{ij} \epsilon_{ijn} \frac{i}{2q_n} < j_i^5, T_{0j} > |_{\omega=0} , \quad (43)$$

(no summation over index  $n$ ). The argumentation is based on the well known analogy between the vector potential of a gauge field  $\mathbf{A}$  and the metric-related vector  $\mathbf{g}$ , where  $g_i \equiv g_{0i}$ , or  $ds^2 = dt^2 + 2g_i dt dx^i + dx_i^2$ . In the rest-frame of the fluid but in the background of the gravitational potential  $\mathbf{g}$  we have for the 4-velocity of the liquid  $u_\mu = (-1, \mathbf{v}) = (-1, \mathbf{g})$ . Therefore the "gravi-magnetic field"  $\mathbf{B}_g \equiv \text{curl} \mathbf{g}$  and we find, indeed, a complete analogy between the coefficient  $\sigma_B$ , see Eq. (31), describing the chiral magnetic effect and the coefficient  $\xi_\omega$ , see Eqs. (42) and (43), describing the "chiral gravi-magnetic effect", or the chiral vortical effect as we call it.

Let us recall the reader that the non-renormalization theorem of Sect. 3.1 fixes the chiral vortical effect up to a temperature -dependent term, proportional to  $T^2$  which can be evaluated at the vanishing chemical potential  $\mu = 0$ , see Eq.(23). We turn now to the problem of evaluating this missing term. In view of Eq. (43) we are interested then in the following term in the effective action:

$$S_{eff} = i\xi_\omega \int d^3x \epsilon_{ijk} A_i^5 \partial_j g_k \equiv iT^2 C_\omega \int d^4x a_i \partial_j b_k , \quad (44)$$

where  $a_i \equiv A_i^5$  is the gauge field coupled to  $j_i^5$ ,  $Tb_i \equiv g_{0i}$  is a component of the metric. We consider linearized gravity and the action is to be invariant under gauge and diffeomorphic transformations

$$a_i \rightarrow a_i + \partial_i \alpha, \quad b_i \rightarrow T(\nabla_i \epsilon_0 + \nabla_0 \epsilon_i) \quad (45)$$

where  $\alpha$  is an arbitrary function,  $\epsilon_\mu$  is the diffeomorphism parameter,  $x_\mu \rightarrow x_\mu + \epsilon_\mu$ .

The Lagrangian density corresponding to the action (44) describes a 3d non-diagonal mass term mixing wave functions of the vectors  $a_i$  and  $b_i$ . The action (44) is gauge and diffeomorphic invariant, as it should be. However the mass term itself is not. This observation allows eventually to prove cancellation of a large class of radiative corrections [19]. The 3d nature of the action (44) is crucial for the proof. Note that although we started with a 4d gauge theory, reduction to a sum over a sequence of 3d theories is inherent to the problem since at finite temperature any 4d field theory reduces to a sum over Matsubara frequencies, with each frequency corresponding to a 3d theory.

The analysis of the radiative corrections to the 3d topological term (44) echoes the proof of non-renormalizability of a pure gauge-boson topological mass given about 30 years ago [21]. In that case the topological mass looks as [37]:

$$S_{eff}^{gauge} = im_g \int d^3x \epsilon_{ijk} a_i \partial_j a_k . \quad (46)$$

This topological mass arises on one-loop level in 3d gauge theories. The simplest Lagrangian of the matter field looks as

$$L_m = \bar{\psi}(D_\mu \gamma_\mu - m_0)\psi, \quad (47)$$

where  $D_\mu$  ( $\mu = 0, 1, 2$ ) are covariant derivatives. The one-loop contribution is given by

$$(m_g)_{one-loop} = \frac{q^2}{4\pi} \frac{m_0}{|m_0|}, \quad (48)$$

where  $q$  is the charge of the fermion. In higher orders of perturbation theory one-loop fermion graph with a few photon exchanges arise. One can integrate first over the (massive) fermion and reduce the graph to a  $n$ -photon effective vertex. In the momentum space, it is denoted as  $\Gamma_{\mu_1 \dots \mu_n}^{(n)}(q_1, \dots, q_n)$  where  $q_i$  are photon momenta and the overall  $\delta$ -function is factored out. The vertex is a function of  $(n-1)$  independent momenta,  $q_1, \dots, q_{n-1}$ . Take now the limit of all the momenta ( $q_1, \dots, q_{n-1}$ ) small. Then it is straightforward to prove that the effective vertex is to vanish in the limit of any independent momentum zero. In other words,

$$\Gamma^{(n)}(q_1, \dots, q_n) = O(q_1 \cdot q_2 \cdot \dots \cdot q_{n-1}). \quad (49)$$

This relation is sufficient to prove that all the contributions to topological mass, beginning with two loops, vanish. Indeed, one can always choose the momenta corresponding to the external photon legs to be included into momenta ( $q_1, \dots, q_{n-1}$ ) which are small. The one-loop graph is exceptional in this sense since there is only a single independent photon momentum and  $\Gamma^{(1)}(q_1) = O(q_1)$ .

Let us come back to discussion of the topological mass (44) relevant to the vortical effect. It is determined by the correlator (43). Let us split the momentum-density operator into fermionic and gluonic parts:

$$T_{0j} = (T_{0j})_{fermionic} + (T_{0j})_{gluonic}. \quad (50)$$

As far as we keep only the fermionic part the main idea, that only 3d one-loop graphs can contribute to (44), remains the same. There is, however, an important change concerning infrared behavior of higher-loop graphs. Amusingly, the masslessness of the fermion in the original 4d field theory does not matter since in the 3d projection the fermions do have non-vanishing masses,  $m_f^2 = 4\pi^2(n + 1/2)^2$  where  $n$ , ( $n = 0, 1, 2, \dots$ ) enumerates Matsubara frequencies. The actual subtle point is that in non-Abelian 3d theory higher loops generically diverge badly in the infrared. According to Ref. [19] the infrared cut off is still provided by the non-perturbative gluon mass emerging [38] at finite temperatures. The effective gluon mass is of order  $(m_g^2)_{eff} \sim g_{Y-M}^4(T)T^2$  and at external momenta much smaller than this effective mass one can expect that higher-loop contribution to the topological mass still vanishes. Finally, the evaluation of one-loop fermionic contributions to the topological mass (44) is more involved technically than in case of (46). Each 3d theory corresponding to the  $n$ -th Matsubara frequency does contribute to (44). Indeed, according to (48) the one-loop contribution does not



disappear with the growing fermion mass. As a result, one comes [19] to a divergent sum. The standard  $\zeta$ -function regularization provides the final answer for the temperature-dependent vortical effect:

$$C_\omega^{ferm} T^2 = -T^2 \sum_{m=1}^{\infty} m \rightarrow \frac{1}{12} T^2, \quad (51)$$

where  $C_\omega^{ferm}$  is the contribution of the fermionic loop to  $C_\omega$ .

So far we discussed fermionic part of  $T_{0j}$ , see (50) and argued that its contribution is exhausted by one-loop graphs. The argument does not apply, however, to the gauge-field part,  $(T_{0j})_{gluonic}$ . It does generate a calculable two-loop radiative correction to the  $T^2$  term in Eq. (9), see [20,19]. We should not feel disappointed about this lack of complete cancellation of higher loops. Indeed, although the Adler-Bardeen theorem is commonly referred to as a proof of non-renormalization of the anomaly, it is known since long (see, e.g., [39]) that two-loop graphs corresponding to rescattering of gauge field do not vanish in fact. The proof of non-renormalizability of the ChME in Sect. 3.1 avoided this problem only because we treated the electromagnetic field as external (not dynamical). Note also that the  $T^2$  term in the chiral vortical effect is present only in case of singlet currents which are anomalous anyhow and the status of the ChME is not so clear. Its evaluation, however, is an amusing demonstration that dissipation-free processes generically are not suppressed at all at high temperature.

#### 2.4 Non-renormalization theorems in effective field theories

Here we develop another approach to the chiral effects based on an effective theory following mostly Ref. [24]. The basic idea is to treat chemical potentials as effective couplings. The basic rule can be memorized as an analogy between interaction with charge  $q$  and with chemical potential  $\mu$ :  $qA^\mu \rightarrow \mu u^\mu$ , where  $A_\mu$  is the vector-potential of an external field and  $u^\mu$  is, as usual, the 4-velocity of an element of liquid, see Eq. (10).

Let us first substantiate the rule (10). Chemical potentials are introduced through the effective Hamiltonian:

$$\delta H = \mu Q + \mu_5 Q_5, \quad (52)$$

where  $Q = \int d^3x \psi^\dagger \psi$  and  $Q_5 = \int d^3x \psi^\dagger \gamma_5 \psi$ . Note that this is indeed an effective, not fundamental interaction since any chemical potential is introduced thermodynamically, that is for a large number of particles. However, at least formally the Hamiltonian (52) looks exactly the same as the Hamiltonian for a fundamental (i.e., not effective) interaction of charges with the  $A_0$  gauge field. Thus, we come to the analogy (10) in the particular case  $u^0 = 1, u^i = 0$ . This case was considered in many papers, see in particular [9], and in the equilibrium the analogy between  $A_0$  and  $\mu$  is commonly used nowadays. We have already exploited this connection, see Eq. (29). In this sense the effective theory considered here can be viewed as a simplified version of a much more elaborated scheme outlined in Sect. 2.2.

Treating (52) as a small perturbation we can also fix the corresponding Lagrangian density:

$$S_{eff} = \int dx \left( i\bar{\psi}\gamma^\rho D_\rho \psi + \mu\bar{\psi}\gamma^0\psi + \mu_5\bar{\psi}\gamma^0\gamma_5\psi \right) + S_{int}, \quad (53)$$

where  $D_\mu$  is the covariant derivative in external electromagnetic field and  $S_{int}$  is a fundamental interaction responsible for the formation of the liquid. For our discussion it is crucial that  $S_{int}$  does not induce any anomaly. We can drop then this interaction for our purposes.

So far we considered the whole of the liquid as being at rest and the chemical potential being constant through the whole of the volume. To study hydrodynamics one can use the standard trick of boosting the action into a local rest frame by utilizing the 4-vector  $u^\mu$ :

$$S_{eff} = \int dx \left( i\bar{\psi}\gamma^\rho D_\rho \psi + \mu u_\mu \bar{\psi}\gamma^\mu \psi + \mu_5 u_\mu \bar{\psi}\gamma^\mu \gamma_5 \psi \right). \quad (54)$$

The Lagrangian density (54) coincides in the rest frame with (53) but looks perfectly Lorentz invariant and can be used in covariant perturbative calculations. The boost velocity  $u^\mu$  is treated then as a slowly varying external field, similar to  $A^\mu$ .

The only difference between the effective action (54) and fundamental one is the presence of the terms proportional to the chemical potentials  $\mu_5, \mu$ . The fundamental theory is free from anomalies in the case considered now,  $(E \cdot B) = 0$ . Therefore the only possible source of anomalies in the effective theory are the triangle graphs with  $\mu, \mu_5$  entering the vertices.

The presence of an anomaly in the effective theory can indeed be readily verified. For this purpose one can calculate triangle graphs or use Fujikawa-Vergeles path-integral considerations. According to the latter technique, anomalies emerge due to non-invariance of the path-integral measure under field transformations. Consider the following transformation

$$\psi \rightarrow e^{i\alpha\gamma_5 + i\beta} \psi. \quad (55)$$

Then, by the standard technique one readily finds <sup>6</sup>:

$$\partial_\mu j_5^\mu = -\frac{1}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} \left( \partial^\mu (A^\nu + \mu u^\nu) \partial^\alpha (A^\beta + \mu u^\beta) + \partial^\mu \mu_5 u^\nu \partial^\alpha \mu_5 u^\beta \right) \quad (56)$$

$$\partial_\mu j^\mu = -\frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} \partial^\mu (A^\nu + \mu u^\nu) \partial^\alpha \mu_5 u^\beta. \quad (57)$$

Rewriting Eqs. (56) and (57)

$$\begin{aligned} \partial_\mu \left( n_5 u^\mu + \frac{\mu^2 + \mu_5^2}{2\pi^2} \omega^\mu + \frac{\mu}{2\pi^2} B^\mu \right) &= -\frac{1}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} \partial^\mu A^\nu \partial^\alpha A^\beta \\ \partial_\mu \left( n u^\mu + \frac{\mu\mu_5}{\pi^2} \omega^\mu + \frac{\mu_5}{2\pi^2} B^\mu \right) &= 0. \end{aligned} \quad (58)$$

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<sup>6</sup> We could have defined anomaly in such a way that it does not contribute to  $\partial_\mu j^\mu$ . However, in the presence of both chemical potentials  $\mu$  and  $\mu_5$  there is no physical motivation for such a regularization.

allows for a straightforward comparison with results of the thermodynamic approach, see Eqs. (23), (24). We find out that the effective theory does reproduce anomalous pieces of the transport coefficients obtained earlier in the leading order in the chemical potentials. As for the higher orders in the chemical potential there are apparent differences. Within the effective theory higher orders in the chemical potential belong to higher order perturbative terms. The triangle graphs which determine (58) have of course a very special status. First, they defy conservation of the currents and, second, they do not receive contributions due to the iteration of  $S_{int}$ , because of the Adler-Bardeen theorem.

As for higher in  $\mu, \mu_5$  terms they are infrared sensitive and can be fixed only within a particular infrared-sensitive regularization scheme. Consider for example contribution of order  $\mu^3$  to the chiral vortical effect. It can be estimated as

$$\delta\xi \sim \frac{\mu^2}{2\pi^2} \frac{\mu}{\epsilon_{IR}}, \quad (59)$$

where  $\epsilon_{IR}$  is an infrared cut off in the energy/momentum integration. Eq (59) can hardly be improved within the effective theory. On the other hand, the thermodynamic derivation of Sect 2.3 does fix [10] terms of order  $\mu^3$  in the Landau gauge as:

$$\delta\xi = -\frac{\mu^2}{2\pi^2} \frac{2\mu n}{(\epsilon + p)}. \quad (60)$$

By comparing (59) and (60) we find

$$\epsilon_{IR} \sim (\epsilon + p)/n. \quad (61)$$

Note that the enthalpy  $w = \epsilon + p$  is known to play the role of mass in relativistic hydrodynamics. Thus, the ratio  $\mu \cdot n/(\epsilon + p)$  characterizes the contribution of the energy related to the chemical potential in units of the total energy, as far as  $\mu$  is small. And it is quite natural to have an expansion in this parameter. The expansion coefficients within the effective theory are model dependent, however.

To summarize, the effective hydrodynamic theory defined through the substitution (10) allows for a straightforward and simple evaluation of the chiral effects in terms of anomalies of the effective theory. Terms of lowest order in expansion in the chemical potentials coincide with the results of other approaches. Higher orders, however, are infrared sensitive and model dependent within the effective theory. Within the thermodynamic approach of Sect. 2.1 these terms are apparently fixed by the procedure chosen to integrate the differential equations (21) beginning with small  $\mu$  and keeping the pressure  $P$  constant. The use of the Landau frame is also needed.

Apart from the  $\mu^2$  term the vortical effect has a  $T^2$  contribution, see Sect. 2.3. This term can be evaluated within finite-temperature field theory at  $\mu = 0$ . For this reason the effective in chemical potentials theory introduced above does not help to evaluate the  $T^2$  term.

### 2.5 Concluding remarks

Powerful non-renormalization theorems have been proven in case of chiral magnetic and vortical effects in hydrodynamic approximation. Eventually, all the proofs go back to celebrated field theoretic non-renormalization theorems [8,21]. To bridge them to hydrodynamics, it is crucial that chiral effects can be expressed in terms of certain spatial correlators at frequency  $\omega = 0$ . These static correlators are trivially continued to the Euclidean space. Which means, in turn, that we are dealing with thermodynamic observables. Continuation to the Euclidean space also allows to use the standard technique of Feynman graphs and utilize field-theoretic non-renormalization theorems. We also notice that all the theorems refer to topological terms in action. In other words, the action observes symmetries of the problem considered while the Lagrangian density does not. Chiral anomalies were put into such a context first in Ref. [40]. Moreover, it turns out that it is not crucial whether the corresponding one-loop graphs signify an anomaly or not. Probably, one and the same non-renormalization theorem can be proven either in terms of anomalies or non-anomalous graphs. The topological aspect, however, seems to be an indispensable ingredient to non-renormalizability.

## 3 Hydrodynamic Chiral Effects as Quantum Phenomena

### 3.1 Non-dissipative currents

All the chiral effects which we are considering are non-dissipative. This was demonstrated in the thermodynamic language [14,12,13], see Sect. 2.2. Another line of reasoning is suggested in Ref. [27] and is based on time-reversal invariance. Let us summarize the argumentation <sup>7</sup>. One compares, for example, ordinary electric conductance,  $\sigma_E$  and  $\sigma_B$  associated with the ChME:

$$\mathbf{j}_{el} = \sigma_E \mathbf{E}, \quad \mathbf{j}_{el} = \sigma_B \mathbf{B}.$$

Moreover,  $\sigma_E > 0$  since the work done by external electric field,  $W = \mathbf{j}_{el} \cdot \mathbf{E} > 0$ . Under time reversal,

$$\mathbf{j}_{el}^T = -\mathbf{j}_{el}, \quad \mathbf{E}^T = +\mathbf{E}, \quad \mathbf{B}^T = -\mathbf{B}.$$

If we would try to obtain  $\mathbf{j}_{el}^T = \sigma_E^T \mathbf{E}^T$  by time reversal of the relation  $\mathbf{j}_{el} = \sigma_E \mathbf{E}$  then we would conclude that  $\sigma_E^T = -\sigma_E$  which is in contradiction with the positivity of  $\sigma_E$ . There is no surprise of course in this failure. To the contrary, dissipation is indeed not a time-reversible process. (The time-reversal invariance is manifested instead in the Onsager relations which imply, in particular,  $\sigma_E > 0$ ). On the other hand, the Hall relation  $\mathbf{j}_{el} \sim \mathbf{E} \times \mathbf{B}$  is time-reversal invariant and this is possible only if there is no dissipation. Our chiral magnetic effect is of the same type as the Hall conductivity and, therefore,

<sup>7</sup> The author is thankful to L. Stodolsky for a detailed discussion of the subject.

cannot be accompanied by dissipation because of the time-reversal invariance of theories considered. Note that there are no such symmetry-based arguments for, say, superfluid current. The viscosity can be only dynamically suppressed in that case and according to the modern views [41] there is a universal lowest bound on the shear viscosity,  $\eta/s \geq 1/4\pi$  where  $s$  is the entropy density. According to the logic outlined above no similar bound can exist for dissipation associated with the chiral magnetic effect since dissipation is forbidden by symmetry considerations.

Furthermore, dissipation-free processes are usually quantum phenomena and in this section we discuss the ChME from this point of view. In fact, it is rather an open-end discussion since not much known yet about the microscopic picture of the ChME in the hydrodynamic approximation.

There is no doubt that the ChME is rooted in the loop graphs of the underlying theory and is represents a macroscopic manifestation of quantum phenomena. Calculation of a standard Feynman graph gets related to thermodynamics by a simple trick of identifying a constant piece in an external gauge field with the chemical potential,  $A_0 \approx \mu$ . Let us explain this point in more detail. The generating functional  $W(\text{sources}(x))$ , see Eq. (28), contains in fact a piece,  $W_{anom}^{(1)}(\text{sources}(x))$  reproducing the chiral anomaly. Moreover,  $W_{anom}$  is uniquely fixed by the requirement that it does reproduce the anomaly within the formalism of Sect. 2.2. It turns out that  $W_{anom}$  in the static case considered can be written in a local form [12]. Explicitly to first order in derivatives:

$$W_{anom}^{(1)} = \frac{C}{2} \int d^3x \sqrt{g_3} \left( \frac{A_0}{3T_0} \epsilon^{ijk} A_i \partial_j A_k + \frac{A_0^2}{6T_0} \epsilon^{ijk} A_i \partial_j a_k \right) , \quad (62)$$

where the constant  $C$  determines the anomaly and further notations are specified in Sect. 2.2.

Let us emphasize that Eq. (62) is a pure field-theoretic input valid in the exact chiral limit. However, once we identify  $A_0$  with a macroscopic quantity, chemical potential, the action (62) determines macroscopic motions. Moreover, it is quite obvious that two terms in the right-hand side of Eq. (62) do reproduce the chiral magnetic and chiral vortical effects, respectively. Comparison of (62) with (31) and (43) helps to check this.

As is emphasized above, Eq. (62) is valid in exact chiral limit. If we would decide to estimate the effect of chiral symmetry violations, say, through finite fermionic masses, we have to address the field-theoretic calculations anew. In particular, the rate of production of massive fermions in parallel constant electric and magnetic fields,  $E_z, B_z$  is given by the equation [42]:

$$\frac{dN_5}{d^3x dt} = \frac{q^2 B_z E_z}{2\pi^2} \exp \left( - \frac{\pi m_f^2}{E_z} \right) , \quad (63)$$

which is replacing Eq (26). We see that if we tend  $E_z \rightarrow 0$  now we would not get any ChME, compare Eq. (27). Therefore, the formal static hydrodynamic limit is replaced for small fermionic masses by:

$$(\omega \equiv 0, q_i \rightarrow 0) \rightarrow (|q_i| \gg \omega, \omega \gg m_f) , \quad (64)$$

and phenomenological implications of this modification are to be considered within a particular framework.

Quantum description of the chiral magnetic effect has another aspect which can be illustrated on the example of chiral fermions interacting with external magnetic field [6,2]. Namely, we should be able to obtain the same current (1) by evaluating the matrix element

$$j_\mu^{el} = q \langle \bar{\psi}(x) \gamma_\mu(x) \psi(x) \rangle , \quad (65)$$

where the averaging is over the thermodynamic ensemble. The simplest set up is non-interacting fermions and a non-vanishing chemical potential  $\mu_5$ . In this case  $\psi(x)$  represents solutions of the Dirac equation

$$(i\gamma^\mu D_\mu + \mu_5 \gamma^0 \gamma^5) \psi(x) = 0 , \quad (66)$$

where  $D_\mu = \partial_\mu - iA_\mu$  and the vector-potential  $A_\mu$  corresponds to a constant magnetic field. The thermodynamic ensemble is described by the ideal gas of massless Fermi particles. Explicit calculation turns to be feasible and the final result agrees with (1). We will give some details of the calculation in Sect. 3.4. Note that, at least from the technical point of view, this coincidence is not trivial at all. Indeed, Eq. (1) is based entirely on the evaluation of the anomalous triangle graph over perturbative vacuum state. The magnetic field is treated perturbatively. On the other hand, the matrix element (65) is saturated by the fermionic zero modes [6] which one finds explicitly, accounting for the magnetic field to all orders.

What is lacking in case of the ChME in hydrodynamic approximation is a microscopic calculation similar to the direct evaluation of (65) for non-interacting particles just outlined. What makes such a calculation especially desirable is the observation that all the non-renormalization theorems reviewed in Sect. 2 do not indicate any crucial dependence on temperature. On the other hand, well known non-relativistic analogues, like the Hall conductivity or ordinary superfluidity do exhibit sharp dependence on the temperature.

In the rest of this section we discuss microscopical picture for the chiral effects on the only example available so far [30], that is the case of superfluid.

### 3.2 Low-dimensional defects

A novel point brought by considering superfluid is the crucial role of low-dimensional defects, or singularities of hydrodynamic approximation. Such defects were considered in fact in many papers for various reasons, see in particular [28,29,17,43]. In case of rotating superfluid such defects have been known since long [44].

Consider first the chiral vortical effect. The crucial point is that the velocity field of the superfluid is known to be potential:

$$\mathbf{v}_s = \nabla \varphi , \quad (67)$$

and, naively, the vorticity vanishes identically since  $\text{curl} \mathbf{v}_s = 0$ . If this were true, the chiral current (7) would disappear. But it is well known, of course, that

the angular momentum is still transferred to the liquid through vortices [44]. The potential is singular on the linear defects, or vortices. The vortex is defined through circulation of velocity:

$$\oint \mathbf{v}_s d\mathbf{l} = 2\pi k, \quad (68)$$

where  $d\mathbf{l}$  is an element of length and  $k$  is integer. The quantization condition (68) follows from the interpretation of  $\varphi$  as the phase of a wave function of identical particles. Eqs. (68), (67) imply that the velocity is singular at the "beginning of the coordinates". In three dimensions the singularity occupies a line, which can be called a defect of lower dimension, or vortex. The energy of the vortex is logarithmically divergent,  $E_{vortex} \sim l \ln(l/a)$  where  $l$  is the length of a (closed) vortex and  $a$  is of order distance between the constituents.

Note that the velocity  $v_s$  according to (68) falls off as a function of distance  $r$  to the singularity,  $v_s = k/r$  while for a rotating solid body the velocity, to the contrary, would grow with distance to the axis of rotation,  $v = |\boldsymbol{\Omega}|r$  where  $\boldsymbol{\Omega}$  is the angular velocity of rotation. Therefore, at first sight, distribution of velocities inside a rotating bucket with a superfluid is very different from ordinary liquid. This is not true, however, as far as the angular velocity is large enough. In this case, there are many defects and the distribution of velocities, once averaged over defects, is the same as for ordinary liquid. The proof [44] is based on the observation that thermodynamic equilibrium is determined by minimizing  $E_{rotation} = E - \mathbf{M} \cdot \boldsymbol{\Omega}$  where  $\mathbf{M}$  is angular momentum and the averaged (over defects) velocity is uniquely determined by this condition.

To summarize, vortices in superfluid represent a well-known example of lower-dimensional defects. Although locally, or at small distances, defects look very different from the hydrodynamic picture, the thermodynamic results can be restored upon averaging over a large number of defects.

### 3.3 Relativistic superfluidity

To consider vortex solutions in detail we need explicit examples of dynamical systems which exhibit relativistic superfluidity. The simplest and best understood example [45] of this type seems to be the pion medium at zero temperature and non-zero isospin chemical potential  $\mu_I$ . The system is described by the chiral Lagrangian;

$$L = \frac{1}{4} f_\pi^2 \text{Tr}[D^\mu U (D_\mu U)^\dagger], \quad (69)$$

where  $U$  are  $2 \times 2$  unitary matrices, functions of the pionic fields, see, e.g., [46]. Moreover, the chemical potential  $\mu_I$  is switched on through the covariant derivatives,  $D_0 U = \partial_0 U - \frac{\mu_I}{2} [\tau_3, U]$ ,  $D_i U = \partial_i U$ . To have the chiral current conserved we consider massless quarks. Then the chiral symmetry is spontaneously broken. In the common case of vanishing chemical potential the residual symmetry (realized linearly) is  $SU(2)_{L+R}$ . Furthermore, at non-zero  $\mu_I$  this symmetry is

broken to  $U(1)_{L+R}$ . The proof is straightforward. Namely, the potential energy corresponding to (69) equals to

$$V_{eff}(U) = \frac{f_\pi^2 \mu_I^2}{8} \text{Tr}[\tau_3 U \tau_3 U^\dagger - 1], \quad (70)$$

and the minima of that potential can be captured by substitution  $U = \cos \alpha + i(\tau_1 \cos \phi + \tau_2 \sin \phi) \sin \alpha$ :

$$V_{eff}(\alpha) = \frac{f_\pi^2 \mu_I^2}{4} (\cos 2\alpha - 1) \quad (71)$$

and for the minimum one readily obtains  $\cos \alpha = 0$ . Then, depending on the sign of  $\mu_I$ , squared mass of  $\pi^+$  or  $\pi^-$  state becomes negative and the corresponding field is condensed. This means that the vacuum is described by  $U = i(\tau_1 \cos \phi + \tau_2 \sin \phi)$  instead of the standard, i.e.  $\mu = 0$  vacuum  $U = I$ . There emerges a new order parameter  $\langle \bar{u} \gamma_5 d \rangle + h.c. = 2 \langle \bar{\psi} \psi \rangle_{vac} \sin(\alpha) = 2 \langle \bar{\psi} \psi \rangle_{vac}$ . The system is thus a charged superfluid. It should be noted, that the degeneracy with respect to the angle  $\phi$  above indicates that it can be identified as a 3d Goldstone field. In addition, there are two massive modes.

Because of the presence of a 3d Goldstone mode the superfluidity criterion (32) is satisfied:

$$\lim_{\mathbf{q} \rightarrow 0} \int d^3x \exp(i\mathbf{q} \cdot \mathbf{r}) < T_{0i}(\mathbf{x}), T_{0k}(0) > = \mu^2 \mathbf{q}_i \mathbf{q}_k / \mathbf{q}^2. \quad (72)$$

Explicit evaluation of this correlator is based on the Josephson equation:

$$\partial_0 \phi = \mu \quad (73)$$

which is satisfied now as an equation of motion following from (69). For  $\mu = \text{const}$  Eq. (73) can be interpreted as condensation of  $\partial_0 \phi$ , similar to the standard Higgs condensation but violating the Lorentz invariance [9,47].

As is argued, e.g., in [9],  $\partial_\mu \phi$  can be identified with non-normalized superfluid velocity. The vortex configuration is in principle determined by the value of the angular velocity of rotation of the liquid [11]. We address a generic case, when the quantum of circulation,  $n$  is rather high to consider defects thermodynamically but not high enough to ruin the superfluidity. As is known, an energetically preferable configuration is the uniform distribution of vortices with  $n = 1$ . Nearby any given vortex the Goldstone field is given by [47]:

$$\phi = \mu t + \varphi, \quad (74)$$

We will assume that vortices are well separated,  $\delta x \gg a$ , calculate the current for a single vortex and then sum it over all vortices, that is simply multiply by  $n$ . The effective Lagrangian for the interaction of fermions with the scalar field  $\phi$  (we will limit ourselves to the case of single fermion, and then sum up the result over colors and flavors) looks as

$$L = \bar{\psi} i(\partial_\mu + i\partial_\mu \phi) \gamma^\mu \psi. \quad (75)$$



Indeed because, of the Josephson equation (73) we reproduce the standard chemical potential term. Other components complete this term to a formally Lorentz invariant interaction, compare Eq. (10).

Using standard methods of evaluating the anomalous triangle diagrams, see, e.g., Sect. 2.4, one obtains for the axial current:

$$j_\mu^5 = \frac{1}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} \partial^\nu \phi \partial^\alpha \partial^\beta \phi, \quad (76)$$

This current seems to vanish identically. However, for the vortex field,  $\phi = \mu t + \varphi$ ,  $[\partial_x, \partial_y]\phi = 2\pi\delta(x, y)$  and:

$$(j_3^5)_{vortex} = \frac{\mu}{2\pi} \delta(x, y) . \quad (77)$$

The total current, or the sum over the vortices equals to

$$j_3^5 = \int d^2x j_3^5 = \frac{\mu}{2\pi} n . \quad (78)$$

It is worth noting that actually  $n \sim \mu$  and the current (78) is quadratic in the chemical potential  $\mu$ , as it should be.

So far we considered the chiral vortical effect. To evaluate the chiral magnetic effect, consider a charged superfluid and turn on a magnetic field. Then magnetic field would stream into tubes, or Abrikosov vortices. The vortex profile could be found by accounting for the finite photon mass. The chiral current can be obtained then by substituting the vortex configuration to the Dirac equation and solving it for the modes. The current is concentrated on the vortex center. In the hydrodynamic approximation we are considering the magnetic field is singular and the Dirac equation is poorly defined in this sense. However, using index theorems it is possible to evaluate number of zero modes, and the zero modes saturate the chiral current. We will give more details in the next subsection.

### 3.4 Zero modes

We now proceed to microscopic picture based on the zero modes <sup>8</sup>. The Hamiltonian has the form :

$$H = -i(\partial_i - i\partial_i\phi)\gamma^0\gamma^i + m\gamma^0 \quad (79)$$

and the Dirac equation decomposes as:

$$-H_R\psi_L = E\psi_L \quad (80)$$

$$H_R\psi_R = E\psi_R, \quad (81)$$

---

<sup>8</sup> This subsection is of rather technical nature and can be considered as an appendix. Moreover, the presentation is close to that of Ref. [17], with a substitution  $A_i \rightarrow \partial_i\phi$ .

where  $H_R = (-i\partial_i + \partial_i\phi)\sigma_i$ . Hence, any solution  $\psi_R$  of  $H_R\psi_R = \epsilon\psi_R$  generates both a solution with energy  $E = \epsilon$ ,  $\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$  and a solution with  $E = -\epsilon$ ,  $\psi = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix}$ .

Because of the invariance with the respect to translations in  $z$  direction, we decompose using the momentum eigenstates  $-i\partial_3\psi_R = p_3\psi_R$ . For each  $p_3$ ,

$$H_R = p_3\sigma^3 + H_\perp \quad (82)$$

$$H_\perp = (-i\partial_a - \partial_a\phi)\sigma^a, \quad a = 1, 2. \quad (83)$$

Notice that  $\{\sigma^3, H_\perp\} = 0$ . This means that if  $|\lambda\rangle$  is an eigenstate of  $H_\perp$  with non-zero eigenvalue  $\lambda$ ,  $H_\perp|\lambda\rangle = \lambda|\lambda\rangle$ ,  $\sigma_3\lambda\rangle$  is an eigenstate with eigenvalue  $-\lambda$ . are of the form  $|\lambda\rangle$ ,  $|\lambda\rangle = \sigma_3|\lambda\rangle$ , with  $\lambda > 0$ . Also,  $\sigma_3$  maps zero eigenstates of  $H_\perp$  to themselves, so that all the eigenstates of  $H_\perp$  can be classified with respect to  $\sigma_3$ .

We can now express eigenstates of  $H_R$  in terms of eigenstates of  $H_\perp$ . Since  $[H_R, H_\perp^2] = 0$ ,  $H_R$  will only mix states  $|\lambda\rangle$ ,  $|\lambda\rangle$ . For  $\lambda > 0$ , one can write,

$$\psi_R = c_1|\lambda\rangle + c_2\sigma_3|\lambda\rangle. \quad (84)$$

Solving equations (82) we find for the eigenvalues of energy:

$$\epsilon = \pm\sqrt{\lambda^2 + p_3^2}.$$

This means that every eigenstate of  $H_\perp$  with eigenvalue  $\lambda > 0$  produces two eigenstates of  $H_R$ , while zero modes of  $H_\perp$ ,  $|\lambda = 0\rangle$  are eigenstates of  $H_R$  with eigenvalues:

$$\epsilon = \pm p_3, \quad (85)$$

where the sign corresponds to  $\sigma^3|\lambda = 0\rangle = \pm|\lambda = 0\rangle$ .

Therefore, the zero modes of  $H_\perp$  are gapless modes of  $H$ , capable of traveling up or down the vortex, depending on the sign of  $\sigma_3$  and chirality. These are precisely the carriers of the axial current along the vortex. Let  $N_\pm$  be the numbers of zero modes which are eigenstates of the matrix  $\sigma^3$  with eigenvalues  $\pm 1$ , respectively. Consider zero mode of  $H_\perp$ ,  $|\lambda\rangle = (u, v)$ , where  $u$  and  $v$  are c-functions satisfying

$$\mathcal{D}v = 0, \quad \mathcal{D}^\dagger u = 0. \quad (86)$$

Here

$$\mathcal{D} = -i\partial_1 - \partial_2 - (\partial_1\phi - i\partial_2\phi). \quad (87)$$

Define  $N_+ = \dim(\ker(\mathcal{D}^\dagger))$ ,  $N_- = \dim(\ker(\mathcal{D}))$ , and

$$N = \text{Index}(H_\perp) = N_+ - N_- = \dim(\ker(\mathcal{D}^\dagger)) - \dim(\ker(\mathcal{D})) \quad (88)$$

Note that  $H_\perp$  is an elliptic operator. Its index has been computed within various approaches in papers [48]. In our case the index is given by

$$N = \frac{1}{2\pi} \int dx_i \partial_i \phi = n. \quad (89)$$

Moreover, for the most important case  $n = 1$  the zero mode is easy to construct, see, e.g., Ref. [30]. The result (89) can be also obtained starting from the well known case of magnetic field parallel to z-axis and uniform in that direction. In the latter case the index is given by

$$N = \frac{e}{2\pi} \int dx_i A_i = \frac{e}{2\pi} \int d^2x B_z \quad (90)$$

and by substituting  $eA_i \rightarrow \partial_i \phi$  we arrive at (89). Note, however, that in case of superfluid, which we discuss here, the index is an integer, whereas for non-superconducting case the flux is not quantized and the left-hand side of Eq. (90) is to be understood as the integer part of the right-hand side.

We now proceed to the computation of the fermion axial current at a finite chemical potential  $\mu$ . The axial current density in the third direction is given by:

$$j_5^3(x) = \bar{\psi}(x) \gamma^3 \gamma^5 \psi(x) = \psi_L^\dagger \sigma^3 \psi_L(x) + \psi_R^\dagger \sigma^3 \psi_R(x) \quad (91)$$

We are interested in the expectation value of the axial current along the vortex,  $j_5^3 = \int d^2x \langle j_5^3(x) \rangle$ . At finite chemical potential, we have:

$$\begin{aligned} \langle j_5^3(x) \rangle &= \sum_E \theta(\mu - E) \psi_E^\dagger(x) \gamma^0 \gamma^3 \gamma^5 \psi_E(x) = \\ &= \sum_\epsilon (\theta(\mu - \epsilon) + \theta(\mu + \epsilon)) \psi_{R\epsilon}^\dagger(x) \sigma^3 \psi_{R\epsilon}(x) \end{aligned} \quad (92)$$

Here,  $\theta(\mu - E)$  is the Fermi-Dirac distribution (at zero temperature),  $\psi_E$  are eigenstates of  $H$  with eigenvalue  $E$ ,  $\psi_{R\epsilon}$  are eigenstates of  $H_R$  with eigenvalue  $\epsilon$ . By substitution of the explicit form of  $\psi_{R\epsilon}$  in terms of  $H_\perp$  eigenstates, one obtains:

$$\begin{aligned} \langle j_5^3 \rangle &= \frac{1}{L} \sum_{p_3} \sum_{\lambda > 0} \sum_{s=\pm} (\theta(\mu - (\lambda^2 + p_3^2)^{\frac{1}{2}}) + \\ &+ \theta(\mu + (\lambda^2 + p_3^2)^{\frac{1}{2}})) \langle \psi_R^s(\lambda, p_3) | \sigma^3 | \psi_R^s(\lambda, p_3) \rangle + \\ &+ \frac{1}{L} \sum_{p_3} \sum_{\lambda=0} (\theta(\mu - p_3) + \theta(\mu + p_3)) \langle \lambda | \sigma^3 | \lambda \rangle. \end{aligned} \quad (93)$$

Here  $\lambda > 0$  enumerate eigenstates of  $H_\perp$ , which generate eigenstates of  $H_R$ ,  $\psi_R^\pm(\lambda, p_3)$  with momentum  $p_3$  and eigenvalue  $\epsilon_\pm = \pm \sqrt{\lambda^2 + p_3^2}$ , and  $\lambda = 0$  label the zero modes of  $H_\perp$ . Moreover, the sum over all non-zero eigenstates vanishes, and only zero modes of  $H_\perp$  generate  $j_5^3 \neq 0$ . For the zero modes,  $\langle \lambda | \sigma^3 | \lambda \rangle = \pm 1$ ,

and we obtain:

$$\begin{aligned} j_5^3 &= (N_+ - N_-) \frac{1}{L} \sum_{p_3} (\theta(\mu - p_3) + \theta(\mu + p_3)) = \\ &= n \int \frac{dp_3}{2\pi} (\theta(\mu - p_3) + \theta(\mu + p_3)) = \frac{\mu}{\pi} n \end{aligned} \quad (94)$$

This result is similar to the macroscopic answer (7) for the vortical effect but there is inconsistency of a factor of two.

### 3.5 Concluding remarks

Considering superfluid provides a unique possibility to develop a microscopic picture for chiral hydrodynamic effects. The calculations above demonstrate that the chiral currents are carried by quantum-mechanical zero modes and are indeed dissipation-free. This result is in agreement with the expectations.

However, this explicit example brings also new lessons. First of all, the prediction for the chiral vortical effect is changed by a factor of two and it is instructive to appreciate the reason for this change. Technically, the easiest way to trace the origin of this factor of two is to compare the calculation of the chiral vortical effect in this section with the calculation within effective theories, see Sect. 2.4. In the latter case, the chiral effects are described by anomalous triangle graphs, with vertices proportional to  $\mu u_\mu$  or  $q A_\mu$ . In other words, the chemical potential  $\mu$  plays the role similar to the electromagnetic coupling, or charge  $q$  while the field of fluid velocities,  $u_\mu$  is similar to the electromagnetic field  $A_\mu$ . The triangle graphs for the chiral magnetic and vortical effects looks very similar. The only difference is that vortical effect is quadratic in  $\mu$  while the magnetic effect is linear both in  $q$  and  $\mu$ . Because of quantum mechanics, however, this difference results in a factor of two: in case of the vortical effect the corresponding graph has two identical vertices and this brings a factor of 1/2, as usual. It is this factor which is absent from the calculation of the vortical effect in terms of defects. Indeed, there are two facets of the chemical potential. First, it plays the role of an effective coupling, as we have just explained. And, second, it limits the integration over the longitudinal momentum of zero modes, see Sect. 3.4. In the language of defects, these two roles are not interchangeable and the quantum-mechanical factor of 1/2 of the effective theory is not reproduced.

Capitalizing on this technical explanation, we can say that in terms of defects we have a two-component picture. One component is superfluid with velocity field  $u_\mu$ . The other component are zero modes responsible for the chiral magnetic and vortical effects. The zero modes are having speed of light and, therefore, are not equilibrated with the rest of the liquid. The two-component picture, however, does not necessarily differ in predictions from one-component picture. Indeed, we did reproduce the standard answer in the case of chiral magnetic effect. Technically, the reason is the same as in Sect. 4.1 where we argued (following Ref. [44]) that vortices reproduce on average the velocity distribution of ordinary rotating liquid. Namely, linearity of the ChME effect in the chemical potential

is the same crucial as linearity in the angular momentum of the energy  $E_{rot}$  in case of rotating liquid.

## 4 Conclusions

Theory of the chiral magnetic effect has been developing fast since the papers [2] put it into the actual context of the RHIC experiments. At the beginning, theory was focused on the mechanism of chirality production in heavy ion collisions. Already at this stage one has to turn to hydrodynamics since it describes the bulk of the RHIC data. Since effective chiral chemical potential  $\mu_5$  vanishes on average and fluctuates from event to event in heavy ion collisions, it is mostly physics of fluctuations which—in theoretical perspective—was studied at this stage.

Later, beginning with the paper [10] the interest shifted from phenomenology to more theoretical issues, such as unifying methods of field theory and thermodynamics to get exact results for chiral effects in hydrodynamic approximation. Very recently, we believe, the most exciting development is the emerging proof that chiral magnetic effect in hydrodynamics is a dissipation-free process. Moreover, examples known so far indicate that this ballistic-type of transport is provided by quantum-mechanical zero modes.

All these exciting results are valid, strictly speaking, in the exact chiral limit. This limitation might have rather severe phenomenological implications in case of realistic quantum chromodynamics. One could expect, therefore, in the near future a shift of interest to condensed-matter systems with fermionic excitations and linear dependence of the energy on the momentum. Of course, this classification of theoretical developments into various stages at the very best could be true only in its gross features. Nevertheless, it might be reasonable to present our conclusions within this, oversimplifying scheme.

**Fluctuations of the chiral chemical potential.** Because of the space limitations we did not address the issue of fluctuations in the bulk of the review. If we start with  $\langle \mu_5 \rangle = 0$  the current (1) vanishes. The chiral magnetic effect is still manifested through fluctuations. In particular, the spectrum of hydrodynamic excitations is sensitive to it. The so called chiral magnetic wave [49] corresponds to the following sequence of fluctuations:

$$(\delta n_Q) \rightarrow (\delta j^5) \rightarrow (\delta \mu_A) \rightarrow (\delta j^{el}) \rightarrow (\delta n_Q) . \quad (95)$$

In more detail: first, a local fluctuation of electric charge density induces fluctuation of axial current, see (8). Then the fluctuation of the axial current triggers a local fluctuation of the axial chemical potential. Finally and completing the cycle, the fluctuation of  $\mu_5$  results in a fluctuation of the electrical current, see (1). Thus, there should exist an excitation combining density waves of electric and chiral charges, the chiral magnetic wave.

Another result to be mentioned here is the observation [50] that there is a piece in the correlator of components of electric current which is uniquely

determined in terms of the chiral anomaly squared:

$$F_{zz}(\omega) - F_{xx}(\omega) = \frac{(qB)^2}{4\pi^3} \frac{\omega}{e^{\beta\omega} - 1} , \quad (96)$$

where the  $z$ -axis points in the direction of an external magnetic field and  $F_{ii}(\omega)$  is a correlator of the  $i$ -th components of the electromagnetic current as a function of frequency  $\omega$ , for a precise definition of the correlators see [50].

**Non-renormalization theorems and non-dissipative motions.** The present review is focused on the non-renormalization theorems and non-dissipative, or quantum nature of the chiral effects, see Sect. 2 and Sect. 3, respectively. Non-renormalization theorems were proven within various approaches (thermodynamic, geometric, diagrammatic, effective field theories). There are no doubts that the non-renormalization theorems are valid within the approximations and assumptions made. The basic assumptions are exact chiral limit, hydrodynamic expansion in derivatives and equilibrium. The main message is that chiral currents are dissipation free and there is no suppression at high temperature.

The results reviewed in Sect. 2 imply that the dissipation-free motions considered are rather ballistic transports than superfluid-type phenomena. Indeed, the entropy current associated with the chiral effects is not vanishing, unlike the superfluid case. According to (22), (24)

$$s_\mu \sim \frac{\mu}{T} \mu_5 B_\mu ,$$

where  $s_\mu$  is the entropy current in equilibrium associated with the magnetic field  $B_\mu$  and  $\mu, \mu_5$  are chemical potentials. The entropy current disappears for  $\mu = 0$  but this is actually a kind of misleading. The point is that there is always a flow of degrees of freedom along the magnetic field and in the direction opposite to it. There is a cancellation, in terms of  $s_\mu$  at  $\mu = 0$  while  $\mu \neq 0$  implies that the liquid is charged and there is preferred direction of the flow of degrees of freedom. The electric current (1), on the other hand, counts the total number of degrees of freedom with charge  $\pm q$ .

Moreover, at least in case of noninteracting particles the carriers of the electromagnetic current (1) are identified as quantum-mechanical zero modes  $\psi_0$  of Dirac particles in the external magnetic field  $B$ :

$$D_\mu \gamma^\mu \psi_0 = 0, \quad N_0 \sim |q \cdot B \cdot \mu_5| ,$$

where  $N_0$  is the number of zero modes, see Sect. 3.4 or, e.g., [3] and references therein. Theoretically, no suppression of the current (1) is expected at non-zero temperatures. A straightforward conclusion would be that the number of zero modes does not go down with temperature. At first sight, it looks very unexpected that quantum coherence could persist at high temperature. Let us mention, however, that something similar happens already at  $T = 0$ . Namely, measurements at small lattice spacings  $a$  (such that  $a \cdot \Lambda_{QCD} \ll 1$ ) demonstrate that the number of zero modes survive wild quantum fluctuations which are of order  $|A_\mu^{glue}| \sim 1/a$ . However, the volume  $V_0$  occupied by a zero mode goes to

zero as a power of  $(a \cdot \Lambda_{QCD})$ , for a review see, e.g., [51]. By analogy, one could expect for zero modes at high temperature

$$N_0(T) \approx \text{const}, \quad V_0 \approx (\Lambda_{QCD}/T)^\gamma,$$

where the index  $\gamma \sim (1 - 2)$ , and zero modes at high temperature become a kind of lower-dimensional defects discussed in Sect. 3.3.

Technically, derivation of dissipation-free hydrodynamics from chiral symmetry of field theory is quite straightforward. Consider first confining phase. In field theory, there is spontaneous symmetry breaking and light degrees of freedom are represented by massless Goldstone fields  $\varphi$ . This is field theoretic input. The hydrodynamic output is superfluidity [45] associated with an extra thermodynamic potential (superfluid 3d velocity squared). The route from field theory to hydrodynamics is provided by replacing ordinary time derivative with a covariant one, the chemical potential being a constant part of the gauge field  $A_0$ . As a result, 4d massless Goldstone particle on the field-theoretic side becomes a 3d massless fields (plus the Josephson condition,  $\partial_0 \phi = \mu$ ). Gradient of  $\varphi$ ,  $\nabla \varphi = \mathbf{v}_s$  is identified with (unrenormalized) superfluid velocity, or a new thermodynamic variable. Original fermionic degrees of freedom are counted as constituents of the normal component. Standard relativistic superfluidity is reproduced.

In this review, we are concerned with the case when there is no spontaneous breaking of the chiral symmetry. Still, there are massless particles, the chiral fermions themselves. Field theoretic input is then existence of polynomials in the effective action such that the action observes the symmetries while the density of the action does not respect the symmetries. The bridge to hydrodynamics is provided by the same inclusion of the chemical potential into the covariant derivative. As a result, there are new motions, or currents allowed in the equilibrium (similar to superfluidity). However, because we do not introduce new massless degrees of freedom, the dissipation-free motions are calculable now, and there are no new thermodynamic potentials. (For attempts to introduce chiral superfluidity in direct analogy with the ordinary superfluidity, i.e. through postulating a new light scalar see Refs [52]).

In this sense, analogy between the ChME and Casimir forces, (for review and references see, e.g., [53]) seems to be more relevant. One considers first interaction of two fluctuating dipoles (atoms) with electric polarizabilities  $\mathbf{p}_{1,2}(\omega) = \alpha_{1,2}(\omega) \mathbf{E}(\omega)$ . At large distances retardation becomes important and the van der Waals potential is replaced by the Casimir-Polder static energy:

$$V_{CP} = - \frac{23}{4\pi} \frac{\alpha_1(0)\alpha_2(0)}{r^6} \frac{\hbar c}{r}$$

Note similarity with evaluation of the ChME: in both cases static quantities are considered. However, accounting for time dependence in intermediate state is crucial. The Casimir-Polder potential refers to interaction of point-like sources due to two-photon exchange. Macroscopic interaction arises upon averaging over many point-like sources. Probably the best known example of this type is the

force  $F$  acting on a unit area  $A$  of two conducting plates at distance  $a$ ;

$$\frac{F}{A} = -\frac{\pi^2 \hbar c}{240 a^4} .$$

This is a macroscopic force of quantum origin, calculable from first principles. Similarly, the anomaly derived first for point-like particles becomes a macroscopic chiral magnetic effect upon averaging of a two-fermion exchange over many centers.

**Towards condensed-matter applications.** A specific feature of the ChME is that even at equilibrium there is a non-vanishing current while Casimir forces are static. This difference is entirely due to the fact that chiral particles are massless and cannot be "stopped". Indeed the Casimir-Polder potential above is due to polarizabilities. As far as a constituent is massive we can have static magnetic dipole  $\mathbf{m} = \alpha_M \mathbf{H}$ . However, for a massless, chiral particle there can be no static magnetic moment and, instead, we get a current, see (1).

Already from this simple reasoning we can conclude that transition to chiral effects is singular. If a small mass  $m_f \neq 0$  is introduced the life-time  $T$  of the current (1) is finite and there is no motion in equilibrium,  $t \rightarrow \infty$ , see Sect. 3.1. Whether

$$T \sim \frac{1}{m_f} \text{ or } T \sim m_f^2/\mu_5 ,$$

or else, remains, to our knowledge, an open question. The answer might depend on details of experiment

It seems natural to assume that the proof of the non-dissipative nature of the chiral effects would be generalized to the case of condensed matter systems, like, say, Weyl semimetals with chiral spectrum of excitations

$$\epsilon_f = v k_f ,$$

where  $\epsilon_f, k_f$  are energy and momentum of fermionic excitation. Indeed, many consequences from the chiral anomaly in relativistic field theory have close parallels in condensed-matter systems, see, in particular [54]. From the point of view of applications the condensed-matter systems seem of course more practical and one can start discussing principles of functioning of new kind of devices [55].

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